Category Theory — Activity 4 - Solutions — Name:
Dr. Paul L. Bailey — Friday, September 13, 2019

Definition 1. Let G be a group. The *center* of G is

$$Z(G) = \{z \in G \mid gz = zg \text{ for all } g \in G\}.$$

Problem 1. Let G be a group. Show that $Z(G) \leq G$.

Solution. To show that something is a subgroup, we show **(S0)**, **(S1)**, and **(S2)**.

(S0) Let $g \in G$. By definition, $1 \cdot g = g \cdot 1 = g$. Thus, $1 \in Z(G)$.

(S1) Let $z_1, z_2 \in Z(G)$ and let $g \in G$. Then

$$\begin{aligned} g(z_1 z_2) &= (g z_1) z_2 && \text{by associativity in } G \\ &= (z_1 g) z_2 && \text{because } z_1 \in Z(G) \\ &= z_1 (g z_2) && \text{by associativity in } G \\ &= z_1 (z_2 g) && \text{because } z_2 \in Z(G) \\ &= (z_1 z_2) g && \text{by associativity in } G. \end{aligned}$$

Thus $z_1 z_2 \in Z(G)$.

(S2) Let $z \in Z(G)$ and let $g \in G$. Since G is a group, z^{-1} exists in G, and $z^{-1}z = zz^{-1} = 1$. Since $z \in Z(G)$, we have $gz = zg$. Multiply by z^{-1} on the left to get $z^{-1}gz = g$. Multiply by z^{-1} on the right to get $z^{-1}g = gz^{-1}$. Thus $z^{-1} \in Z(G)$.

□

Definition 2. Let G and H be groups, and let $f : G \to H$. We say that f is a *group homomorphism* if

$$f(g_1 g_2) = f(g_1) f(g_2).$$

We say that f is a *group isomorphism* if f is a bijective homomorphism.

Problem 2. Let $f : G \to H$ be a group isomorphism. Show that $f^{-1} : H \to G$ is a group isomorphism.

Solution. We know that the inverse of a bijective function is a bijective function; it remains to show that it is a homomorphism.

Let $h_1, h_2 \in H$. Since f is bijective, there exist unique $g_1, g_2 \in G$ such that $f(g_1) = h_1$ and $f(g_2) = h_2$. In this case, $g_1 = f^{-1}(h_1)$ and $g_2 = f^{-1}(h_2)$. Now

$$\begin{aligned} f^{-1}(h_1 h_2) &= f^{-1}(f(g_1) f(g_2)) \\ &= f^{-1}(f(g_1 g_2)) && \text{because } f \text{ is a homomorphism} \\ &= g_1 g_2 && \text{because } f^{-1} \text{ is the inverse of } f \\ &= f^{-1}(h_1) f^{-1}(h_2). \end{aligned}$$

□